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Mixing of shear Alfvén wave packets

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Abstract. The propagation of shear Alfvén wave packets in inhomogeneous magnetic fields, at the origin of their distortion, regardless of the occurrence of non-linear coupling, is considered. It is shown that the distortion mechanism can be regarded as mixing process and hence, standard "phase mixing" corresponds to the effect of an "Alfvénic" shear flow while enhanced dissipation at a magnetic X-point corresponds to mixing by an "Alfvénic" strain flow. The evolution of the global wave field is supposed to result from the dynamics of a superposition of wave packets and a kinetic equation for the wave energy is obtained following this eikonal (WKB) description. Since shear Alfvén wave packets experience continuous shearing/straining while transported by an inhomogeneous Alfvénic flow $\mathbf{V}_A$, their mixing process, in physical space, is also a cascade of wave energy in k-space. The wave energy spectrum resulting from this linear mechanism of energy transfer is determined for the special case of waves propagating along chaotic magnetic field lines, the analog of a chaotic mixing process. The latter follows a k^{-1} power-law, in the energy conserving range in k space.

Key words. magnetohydrodynamics(MHD) – waves – Sun:magnetic fields

1. Introduction

Phase mixing, the propagation and enhanced dissipation of shear Alfvén waves, remain active areas of research in studies of the solar corona, both in relation with its heating mechanism and the acceleration of the fast solar wind; see e.g. reviews on the subjects (Browning 1989; Walsh and Ireland 2003; Ofman 2005) and references within. The reason is that if Alfvén waves are excited and dissipate some of their energy in the solar corona, this contributes to the overall coronal energy budget.

Due to the very low collisionality of the medium, a main difficulty of waves dissipation theories is, however, the justification of how these fluctuations can thermalize, i.e. reach the very small dissipative scale, before they leave the corona. But since the coronal plasma is highly inhomogeneous, it is quite natural to invoke waves interaction with the ambient background as a reason for the creation of fine enough scales in the wave field. It is in this context that, phase mixing of Alfvén waves was proposed, by Heyvaerts and Priest (1983), as an enhanced dissipation mechanism. They considered simple inhomogeneity of the medium, in one direction perpendicular to the unidirectional background magnetic field. The existence of a shear in the Alfvén velocity $\mathbf{V}_A$ results in the creation of progressively smaller transverse scales in the wave field while it propagates and is transported along the unperturbed magnetic field. In this case, the build-up of a diffusive scale is linear in time ($dk/dt \propto cst$), resulting in a dissipation time proportional to $S^{1/3} << S$. Here, S is the large Lundquist number $S \equiv \tau_\eta/\tau_A \gg 1$ of the plasma. The corresponding diffusive scale is $S^{-1/3} \ll 1$. Therefore, the dissipation time is found to me much shorter than the very large diffusion time τ_η, proportional to $S \gg 1$, for the wave to dissipate in the absence of inhomogeneity.

More recent studies have shown that the creation of a dissipative scale can be made exponentially fast ($dk/dt \propto cst.k$), with associated time of the order of $\ln S << S^{1/3} << S$ (Similon and Sudan 1989; Petkaki et al. 1998; Malara et. al 2007). The latter improved efficiency of the dissipation results essentially from the consideration of a more complex magnetic geometry, with the property of local exponential separation or divergence of neighboring magnetic field lines. In this case, the diffusive length scale is proportional to $S^{-1/2}$. Therefore, fast dissipation is possible in three-dimensional chaotic magnetic fields (Similon and Sudan 1989, Petkaki et al 1998, Malara et. al 2000, Malara et. al 2007) and in two-dimensional regular ones (Smith et al. 2007, Malara et al. 2003, Ruderman et. al 1998), provided they possess a certain degree of complexity.

One main goal of this work is to show that the propagation and distortion of shear Alfvén wave packets, in inhomogeneous magnetic fields, can be regarded as a "mix-

ing" effect. As is well known, the kinematics of mixing are governed by the topological properties of the flow. It naturally follows that the distortion of Alfvén wave packets, propagating in complex magnetic fields, is the counterpart of a chaotic mixing process(Ottino 1988).

The interaction among shear Alfvén waves is also a well-studied mechanism of energy transfer to small scales, the non-linear distortion resulting from elastic collisions between counter-propagating wave packets producing a flux of their energy to small dissipative scales. Yet, linear distortion through interaction with the background produces a similar flux. In the following, we neglect nonlinearities in the magnetohydrodynamics equations. The reason is to focus primarily on the spectral properties of this energy transfer for waves propagating in complex magnetic fields.

We obtain a kinetic equation for the wave energy from a representation of the global wave perturbation as an ensemble wave packets (Similon and Sudan 1989). Hence, the wave energy spectrum F(k,t) is the sum of the energy of each individual wave packet, with associated wave vector and position, whose time dependence is given by Hamiltonian ray equations. This WKB method has already been employed by several authors (Petkaki et al 1998, Malara et. al 2000, Malara et. al 2007) to study shear Alfvén waves propagation and dissipation in different types of coronal magnetic structures. Our study is a continuation of these works. In section 4, we determine the transient and stationary energy spectra, F(k,t) and F(k), for the special case of non-interacting Alfvén waves propagating in a synthetic chaotic magnetic configuration with quasi-uniform rate of field lines exponentiation (Similon and Sudan 1989). In section 3, we show the correspondence which exists between the problem of the enhanced dissipation of Alfvén waves, transported by inhomogeneous Alfvénic flows, and the problem of mixing (Ottino 1988), reviewed in section 2. Conclusions are given in the last section of this article.

2. Mixing of a passive tracer

The mixing of a passive field ψ advected by a flow $\mathbf{U}$ is described by the linear advection-diffusion equation

$$\frac{\partial\psi}{\partial t}+\mathbf{U}.\nabla\psi=\eta\nabla^2\psi \tag{1}$$

In absence of flow, $\mathbf{U}=0$, the field ψ only diffuses on a large time scale, for small diffusion coefficient η, because this time is proportional to η^{-1}. If the flow has some spatial variation, i.e. it is inhomogeneous, the advected field is stirred. It then develops increasing gradients, until diffusion takes over and ultimately mixes the field. Because diffusion alone is slow to mix, a stirring flow is required to accelerate the dissipative process. An important issue is therefore to find flows $\mathbf{U}$ that have the property of being good mixers. For the enhanced dissipation of shear-Alfvén waves, this property concerns the Alfvénic flow $\mathbf{V}_A$, which by definition, is responsible for the advection of the wave field. To show this, we first concentrate on two standard and simple flow configurations having their counterpart in the problem of dissipation of Alfvén waves, and compare their mixing efficiency.

Consider the effect of shear flow $\mathbf{U}=(U_0'y,0,0)$,

$$\frac{\partial\psi}{\partial t}+U_0'y\frac{\partial\psi}{\partial x}=\eta(\frac{\partial^2\psi}{\partial^2 x}+\frac{\partial^2\psi}{\partial^2 y}+\frac{\partial^2\psi}{\partial^2 z}) \tag{2}$$

with initial condition $\psi(x,y,t=0)=\psi_0\sin k_0x$, which has no dependence in y. When $\eta=0$, the field is just advected, resulting in $\psi(x,y,t)=\sin k_0(x-U_0'yt)$, i.e an unbounded creation of smaller scales along y, the shear-wise direction. When $\eta\neq 0$, a dissipative cut-off is introduced. In this case, we can look for a solution of the form $\psi=\psi_0 f(t)\sin k_0(x-U_0'yt)$, which gives

$$f(t)=\exp(-\eta k_0^2(t+\frac{U_0'}{3}t^3)), \tag{3}$$

hence, the initial field evolves according to

$$\psi(x,y,t)=\psi_0\sin k_0(x-U_0't)\exp(-\eta k_0^2(t+\frac{U_0'^2}{3}t^3)) \tag{4}$$

From the above solution, we obtain that, for $\eta\ll 1$, the mixing time scale associated with the shear flow is

$$\tau_{shear}\propto(\eta U_0'^2k_0^2)^{-1/3} \tag{5}$$

The associated dissipative length l_d is found from Eq.(2), by balancing the advection term, the second on its left, with the diffusion (along y) term, the second on its right, $U_0'l_d(\psi/L)\sim\eta(\psi/l_d^2)$:

$$l_d\sim(\frac{\eta L}{U_0'})^{1/3} \tag{6}$$

By performing a formal spectral transformation of Eq.(2) it is also found that individual modes evolve according to

$$\frac{\partial\widehat{\psi}_k}{\partial t}+\mathbf{U}_0'k_y\frac{\partial\widehat{\psi}_k}{\partial k_x}=-\eta(k_x^2+k_y^2)\widehat{\psi}_k, \tag{7}$$

with the right-hand side of the above expression representing advection in k-space. This also indicates that a shear flow results in a spectral expulsion towards large absolute value shear-wise wave numbers, corresponding to an increase tilting of the phase front in the flow-wise direction. Hence, the physics behind the accelerated diffusion of the tracer ψ in a shear flow is intuitively analogous to the one of *phase mixing*, which occurs for $\mathbf{V}_A=(V_A'y,0,0)$. To gain a further insight into the problem, let us transform the advection-diffusion equation, by introducing the "shearing coordinates" that follows the streamlines :

$$[x'=x-U_0'yt,y'=y,t'=t] \tag{8}$$

From the chain rule for derivatives :

$$[\frac{\partial}{\partial x}=\frac{\partial}{\partial x'},\frac{\partial}{\partial y}=-U_0't\frac{\partial}{\partial x}+\frac{\partial}{\partial y'},\frac{\partial}{\partial t}=-U_0'y\frac{\partial}{\partial x}+\frac{\partial}{\partial t'}], \tag{9}$$

the advection-diffusion is now described by

$$\frac{\partial\psi}{\partial t}=\eta(\frac{\partial^2}{\partial^2 x}+(-U_0't\frac{\partial}{\partial x}+\frac{\partial}{\partial y'})^2)\psi \tag{10}$$

Keeping only the dominant terms in this equation, leads, after introduction of the new variable $\tau = (U_0'^2 t^3)/3$, to a diffusion equation :

$$\frac{\partial \psi}{\partial \tau} = \eta \frac{\partial^2 \psi}{\partial^2 x} \tag{11}$$

As is well known, there are two self-preserving solutions of a diffusion equation: harmonic and gaussian. Their temporal evolution is different, however, their damping time is identical. Indeed, contrary to the harmonic solution studied above, an initial gaussian packet has its amplitude that decays as $1/(\eta\tau)^{1/2}$, while it spreads. For the phase mixing problem, such power law behavior of localized initial wave-packets, i.e its diffusion along field lines, was noticed and studied by Hood et al. (2002); see also Tsiklauri et al. (2003).

Now let us consider passive advection in a simple strain $\mathbf{U} = (-\alpha x, \alpha y, 0)$ with harmonic initial condition independent of z:

$$\frac{\partial \psi}{\partial t} - \alpha x \frac{\partial \psi}{\partial x} + \alpha y \frac{\partial \psi}{\partial y} = \eta\left(\frac{\partial^2 \psi}{\partial^2 x} + \frac{\partial^2 \psi}{\partial^2 y} + \frac{\partial^2 \psi}{\partial^2 z}\right) \tag{12}$$

A solution of the form

$$\psi = \psi_0 f(t) \sin(k_x(t)x + k_y(t)y) \tag{13}$$

is sought for, and hence,

$$\dot{k_x} = k_x \alpha; \quad \dot{k_y} = -k_y \alpha, \tag{14}$$

$$k_x(t) = k_0 \exp \alpha t; \quad k_y(t) = k_0 \exp -\alpha t, \tag{15}$$

$$\dot{f} = -\eta(k_x^2 + k_y^2) f \tag{16}$$

$$f(t) = \exp[-\frac{\eta k_0^2}{2\alpha}(\exp^{2\alpha t} - 1))] \tag{17}$$

The field amplitude has super-exponential decay in time. It follows that for $\alpha > 0$, the mixing time due to a strain τ_{strain} is reduced compared to τ_{shear}, i.e.

$$\tau_{strain} \propto (1/\alpha) \ln(\alpha/\eta k_0^2), \tag{18}$$

the reason is, that for a strain, the wave vector increases *exponentially* with time and not linearly. And we can anticipate that this situation corresponds to enhanced dissipation of Alfvén waves in the presence of a magnetic X-point configuration with $\mathbf{V}_A = (\alpha x, \alpha y, 0)$. This result remains unchanged for the three dimensional stagnation flow $\mathbf{U} = (\alpha x, \alpha y, -2\alpha z)$. The dissipative scale length for advection and diffusion to balance each other in Eq.(12), is given by

$$l_d \sim \left(\frac{\eta}{\alpha}\right)^{1/2} \tag{19}$$

The general problem of the mixing of a passive scalar, in arbitrary flows, can be approached from a WKB description (Antonsen 1996). This is done, for instance, by introducing a filter, $\widehat{\psi}(\mathbf{k}, \mathbf{x}, t) = \int d^3x \exp[-\mathbf{x}^2/\lambda^2 + i\mathbf{k}(\mathbf{x} - \mathbf{x}')]\psi(\mathbf{x}', t)$, with support $\lambda \ll L$, L being the length scale of variation of $\mathbf{U}$. It is a local Fourier transform in $\mathbf{x}$. Applying this transformation to the advection-diffusion equation (1), we obtain

$$D_t \widehat{\psi} = -\eta k^2 \widehat{\psi} \tag{20}$$

with total derivative $D_t = \partial_t + \dot{\mathbf{x}}\nabla + \dot{\mathbf{k}}\nabla_k$ and $k^2 = k_x^2 + k_y^2 + k_z^2$. Equation (20), which describes transport in $(\mathbf{x}, \mathbf{k})$ space is also known as the kinetic wave equation in plasma physics (Sagdeev and Galeev 1969) and includes non-linear coupling terms on its right-hand side. It is equivalently obtained by use of the Wigner distribution (Bastiaans 1979, Antonsen 1996). The respective characteristics are :

$$\dot{\mathbf{x}} = \mathbf{U} = \nabla_k H \tag{21}$$

$$\dot{\mathbf{k}} = -\nabla(\mathbf{k}.\mathbf{U}) = -\nabla H \tag{22}$$

with $H = \mathbf{k}.\mathbf{U}$. The general solution of the kinetic wave equation (20) can be expressed in term of phase space trajectories $\xi(\mathbf{x}', t)$ and $\kappa(\mathbf{x}', \mathbf{k}', t)$ where $d\xi/dt = \mathbf{v}(\xi, t)$ and $d\kappa/dt = -\nabla \mathbf{v}(\xi, t).\kappa$, and $\xi(\mathbf{x}', 0) = \mathbf{x}', \kappa(\mathbf{x}', \mathbf{k}', 0) = \mathbf{k}'$. The solution is $\widehat{\psi}(\mathbf{x}, \mathbf{k}, t) = \int d^3x' d^3k' \widehat{\psi}(\mathbf{x}', \mathbf{k}', 0)\delta(\mathbf{x} - \xi(\mathbf{x}', t))\delta(\mathbf{k} - \kappa(\mathbf{x}', \mathbf{k}', t)) \exp(-\eta \int_0^t \kappa^2 dt')$.

The physical interpretation of the WKB approach to mixing is clear. It means that the advected field is decomposed into an ensemble of "packets" with length scale, λ, much smaller than the one, L, characterizing the spatial variation of the flow $\mathbf{U}$. Therefore, each packet experiences, along its trajectory, a velocity field $\mathbf{U}$ which is a linear function of the space coordinates :$\mathbf{U}(\mathbf{x}_p(t)) + \partial \mathbf{U}/\partial \mathbf{x} \mid_{\mathbf{x_p(t)}} .\mathbf{x}$, $\mathbf{x}_p(t)$ being the position of the packet at time t. An important quantity, therefore, is the local rate of strain tensor, $\partial \mathbf{U}/\partial \mathbf{x}$, of the flow. To sum-up, the packets are deformed only by the shear and the strain considered below. This brings us to the analogous effect for Alfvén wave packets.

3. Mixing of shear Alfvén waves

The rays equations of optics are $\dot{\mathbf{x}} = \nabla_k \omega$ and $\dot{\mathbf{k}} = -\nabla \omega$, with $\omega(\mathbf{k})$ the wave dispersion relation. The eikonal description of MHD waves was introduced by Weinberg (1962); see also Similon and Sudan (1989), Petkaki et al. (1998), who used it in the context of wave dissipation in the corona. For the non-dispersive shear Alfvén waves, with $\omega = \pm \mathbf{k}.\mathbf{V_A}$, the rays equations read

$$\dot{\mathbf{x}} = \pm \mathbf{V}_A, \tag{23}$$

$$\dot{\mathbf{k}} = \mp \nabla(\mathbf{k}.\mathbf{V}_A). \tag{24}$$

with the energy equation being

$$\dot{e}_\pm = -\eta k^2 e_\pm, \tag{25}$$

The latter is required to study the energetics of the mixing process. The equation of motion Eq.(23) states that shear-Alfvén wave packets propagate along magnetic field

lines so they disperse as the field lines do. It provides information on the precise path along which energy is deposited. The enhanced dissipation time follows from equation (24), giving the rate of stretching of wave packets, i.e. the variation of their wave-vector, when combined with the energy equation. Dispersion and stretching are however related, due to the following consideration of two infinitesimally separated trajectories and the phase gradient, $\nabla\psi \equiv \mathbf{k}$, they generate in an inhomogeneous Alfvénic flow $\mathbf{V}_A$. Indeed, constancy of the phase along a trajectory implies that $\psi(\xi(\mathbf{x}+\mathbf{r},t),t) - \psi(\xi(\mathbf{x},t),t) = \delta\xi(\mathbf{x},t).\nabla\psi(\xi(\mathbf{x},t),t) = const$ with $\delta\xi(\mathbf{x},t) = \delta\xi(\mathbf{x}+\mathbf{r},t) - \delta\xi(\mathbf{x},t)$ being the differential separation vector. Thus, $d(\delta\xi.\mathbf{k})/dt = 0$. According to Eq.(23), the trajectory of a wave packet is $\dot{\xi} = \pm\mathbf{V}_A(\xi,t)$ and therefore $d\delta\xi(\mathbf{x},t)/dt = \pm\delta\xi.\nabla\mathbf{V}_A(\xi(\mathbf{x},t),t)$. Hence, the stretching properties of the Alfvénic flow, represented by Eq.(24) for the wave vector are related to the dispersion of magnetic field lines. Considering the phase mixing situation, with an Alfvénic shear flow $\mathbf{V}_A = V_A' y\mathbf{x}$, $dk_y/dt = k_x V_A'$ with solution $k_y(t) = k_x V_A' t$ and integrating the energy equation, the typical time scale for the phase mixing is found to correspond indeed to τ_{shear}, i.e

$$\tau_{mix} \propto S^{1/3} \ll S \tag{26}$$

with S the Lundquist number $S \gg 1$. In the same way, for the enhanced dissipation of shear Alfvén waves at a magnetic X-point, which corresponds to a an Alfvénic strain flow $\mathbf{V}_A = \alpha x\mathbf{x} - \alpha y\mathbf{y}$, $\alpha \equiv V_A'$, the component of the wave vector in the direction of contraction $\mathbf{x}$, increases now exponentially with time, $\mathbf{k}_x(t) = k_0\exp(\alpha t)$. Integrating the energy equation results in the above mentioned superexponential decay and, therefore, the enhanced dissipation time corresponds to τ_{strain},

$$\tau_{mix} \propto \ln S \ll S^{1/3} \tag{27}$$

This dependance of the dissipation time with Lunquist number, for Alfvénic strain, is also typical of wave damping in complicated magnetic geometries, either if the magnetic field is stochastic (Similon and Sudan 1989) or regular, but possesses separators (Malara et al. 2003; Malara et al. 2007). The physics is however the same, and is related to the divergence of nearby magnetic field lines. In all these cases, the dissipation time of a wave packet weakly depends on the Lunquist number, only as a logarithm, $\tau_{mix} \propto \ln S$, and not as a power law, $\tau_{mix} \propto S^{1/3}$ obtained for unidirectional magnetic field, i.e. standard phase mixing. It is now clear that the propagation of shear Alfvén waves in an inhomogeneous magnetic field is equivalent to a mixing process. Since the kinematics of mixing are governed by the topological properties of the flow, it results that the enhanced dissipation of Alfvén waves propagating in a chaotic magnetic field is equivalent, from the energetics viewpoint, to a chaotic mixing process. The spectral property of the wave energy flux, in this linear process, is the subject of the following section.

4. wave energy spectrum in chaotic magnetic fields

Since within the eikonal formulation the global wave perturbation can be treated as a superposition of wave packets with wave-vector $\mathbf{k}_j$, each carrying an energy $e_j(t)$, we can also write the wave energy spectrum in the following form:

$$F(k,t) = \sum F_j(k,t) = \sum e_j(t)\delta(k - |\mathbf{k}_j(t)|) \tag{28}$$

In the presence of a source of wave energy, continuous shearing/straining experienced by wave packets along their trajectory leads to a cascade of their energy, and hence, of the energy of the global wave perturbation. This cascade can be studied in two different ways. In the first, there is no steady source, but only an initial distribution of wave packets and energy in space. For this initial value problem, the interest is in the subsequent transient evolution of the wave energy spectrum $F_I(k,t)$ with initial condition $F_I(k,t=0) = \sum e_j(t=0)\delta(k - |\mathbf{k}_j(t=0)|)$. As time increases, each wave number k_j increases in time (on average), evolves toward the dissipation range and, therefore, the wave energy decays. A second approach consists in taking into account continuous injection of wave energy given by the ongoing perturbations. Therefore, wave packets and energy are supposed to be continually injected at intermediate k values, around k_0, a typical injection scale, with $L^{-1} \ll k_0 \ll k_d$ and with k_d the very small dissipative cutoff and L the global length scale associated with the magnetic field. In this case, electromechanical energy injection, say the photospheric drive, becomes balanced by heating and dissipation at high k, and a time-averaged stationary spectrum $F_S(k)$ builds-up. We turn to the study of the wave energy spectrum that results from quasi-uniform Alfvénic strain experienced by wave-packets as they propagate in a field with destroyed magnetic surfaces, i.e. along chaotic magnetic field lines. This configuration, which is a paradigm in transport theory, for the calculation of anomalous diffusion coefficients (Rechester and Rosenbluth 1977), was first considered in the context of coronal wave dissipation by Similon and Sudan (1989), although they were only interested in the mixing time scale. Calling ε the input wave energy flux, dimensional analysis yields

$$\varepsilon \propto V_A' k F_S(k), \tag{29}$$

with $F_S(k)$ the stationary wave energy spectrum. Hence,

$$F_S(k) \propto k^{-1} V_A'^{-1} \varepsilon, \tag{30}$$

in the range where dissipation is negligible, i.e. for $k < k_d = (V_A'/\eta)^{1/2}$. This relation (30) for the wave-energy, which is again derived below from a kinetic equation, is the same as the Batchelor's spectrum (Batchelor 1959) for scalar variance, obtained for a flow with uniformly distributed rate of strain, i.e. with chaotic streamlines.

An other derivation of (30) follows from writing down the evolution equation for the wave energy spectrum $F(k,t)$. Taking the time derivative of Eq.(28), we obtain

$$\frac{\partial F(k,t)}{\partial t} = -\eta k^2 F(k,t) + \frac{\partial}{\partial k}[V_A' k F(k,t)], \tag{31}$$

The first term on the right-hand side represents the effect of dissipation, $\dot{e} = -\eta k^2 e$, and vanishes for $\eta = 0$. The second term describes transport in k-space for wave packets that experience uniform rate of strain, $V_A' = const$, i.e. $\dot{k}(t) = V_A' k$. The stationary solution of (31) for $\eta = 0$ is again $F_s(k) \propto k^{-1}$. For $\eta \neq 0$, the stationary solution to Eq.(31), with energy source ϵ added at wave numbers where wave energy is fed into the system, is

$$F_S(k) = \frac{\epsilon}{V_A' k} \exp(-\eta k^2/\alpha) \tag{32}$$

This shows the existence of a gaussian dissipative fall-off, for $F_s(k)$, at large $k > k_d$.

Let us note that while the derivation of the kinetic equation (31) is based on the WKB approximation, the dimensional analysis obtention of the k^{-1} stationary spectrum does not rely on the assumption of a separation of scales. The latter is based only on the fact that the wave energy transfer time is a constant, independent of the scale, given by V_A', the quasi-uniform rate of strain experience by the waves.

Finally, we go back to the initial value problem, for transient forcing. Discarding dissipation in Eq.(31), and changing to the new variables, $\xi = \ln(k)$ and $G = kF(k)$, we can write Eq.(31) as

$$\frac{\partial G}{\partial t} = V_A' \frac{\partial G}{\partial \xi} \tag{33}$$

Therefore, this transport equation reveals that, if an input wave energy pulse, $F_I(k, t = 0)$, is initially centered at intermediate k values, say $k = k_0$, it is then carried *conservatively* at constant speed $\alpha \equiv V_A'$, on a logarithmic scale, until it reaches the dissipative range $k = k_d$. *Only there* it experiences fast damping, and finally converts its energy into heat. This spectral viewpoint also contains the time scale involved before dissipation occurs : it is the travel time in the energy conserving range in k-space, $\tau = (\ln k_d - \ln k_0)/V_A' = (2/V_A')\ln(V_A'/\eta k_0^2) \propto \ln S$, which is τ_{strain}.

Before concluding, few comments are due. First, let us note that the *magnitude* of the energy flux ϵ depends on the amplitude of the drive, but not on its frequency, see e.g. Velli (2003). However, the location of the energy source in k-space depends on the range of driving frequencies. Recall that since the dynamical equations are linear, consideration of a broad frequency drive does not alter the shape of the spectrum in the gap between the largest input wavenumber and the dissipative one. The very existence of such conservative range with neither source nor dissipation is suggested by the following reasoning. Since the wave-number associated with the dissipative scale is very large, $k_d = (V_A'/\eta)^{1/2} \gg 1$, the condition that the latter is also much larger than the largest input wavenumber $k_{\parallel max} = \omega/V_A$, i.e. $k_{\parallel max} \ll k_d$, is equivalent to $\omega\tau_A \ll L(V_A'/\eta)^{1/2} \sim S^{1/2}$. This condition can easily hold for also for waves with $\omega\tau_A \gg 1$ or equivalently, $k_\parallel \gg L^{-1}$, for which the WKB approximation holds. In their original work, Similon and Sudan (1989), were able to relax the condition $k_\parallel \gg L^{-1}$, making it possible to study the behavior of the lower frequency fluctuations and "non-WKB" effects. Their tactic is entirely analogous to the reduced magnetohydrodynamics (RMHD) approximation, although they refer to the so-called "ballooning representation" in fusion studies. By doing so, they only restrict the eikonal representation to the perpendicular direction. Low frequency motion at the boundary of a unidirectional magnetic field is well known to produce turbulence and energy cascade leading to a power-law energy spectrum [see for instance two recent works, Rapazzo et al (2007), Dmitruk et al. (2003) and references therein]. For realistic forcing, i.e. small amplitudes of the Alfvén waves that are launched into the corona, $\delta u \sim 10^{-3} V_A$ (Acton et al. 1981), numerically obtained spectra are generally found to be quite steep. The reason resides in the weakness of the nonlinear interactions. However, since the background large scale field is taken as a constant, only the nonlinearity can there provide the necessary transfer of energy to small dissipative scales. The linear mixing process with its associated energy flux is likely to alter energy transfer rates and hence spectra in inhomogeneous magnetic backgrounds. This is suggested by the numerical studies by Malara et al. (1992), Einaudi et al. (1996), who showed that scattering of Alfvén waves with the background inhomogeneity, for a population of waves initially propagating say, upward, can act as a source for downward-propagating waves, and hence as a trigger for the occurrence of nonlinear interactions. We did not consider such effect in the present work and we are not aware of any MHD turbulence theory that accounts for how the linear and nonlinear mechanisms of energy flux combine together. We find this subject an interesting area for further research.

We finally mention that the mixing effect considered in the present article, is the root of a proposed geometrical interpretation for the consequence of non-linear interactions among counter-propagating wave packets. It is clearly illustrated by Fig.1 in Maron and Goldreich (2001). Indeed, take a sample of originally straight vertical field lines which are perturbed by, say, downward-propagating waves. As a consequence, the field lines wander around each other. Therefore, if a localized wave packet is launched and propagates upward in this braided field, it will be distorted [also compare with Fig.2 in Similon and Sudan (1989) or the figure in Rechester and Rosenbluth (1977)]. After several of these collisions, dissipation takes over and the wave packet mixes in its environment.

5. Conclusions

In this work, we consider the wave energy flux to small scales which results from the distortion of Alfvén waves propagating in an inhomogeneous mean magnetic field $\mathbf{B}_0$. This problem is related to turbulence because the existence of a mean field may provide separation between linear and non-linear time-scales. In an inhomogeneous magnetic field, Alfvén wave packets disperse and are stretched by an Alfvénic flow $\mathbf{V}_A = \mathbf{B}_0/(\mu_0\rho)^{1/2}$, whose streamlines are also the magnetics field lines. In a plasma with uniform density ρ, this flow is incompressible, i.e. $\nabla.\mathbf{V}_A = 0$, and its stretching properties are thus only related to magnetic field lines dispersion. While the wave packets transport their energy at the group velocity $\mathbf{V}_A$, and hence while they disperse, in physical space, as the field lines do, they also experience continuous shearing and straining, i.e. they tend to mix. This gives rise to a cascade of the wave energy in k-space. As is well known for mixing, the efficiency of stretching relies on the local exponential separation of nearby trajectories (Ottino 1988). Since for a continuous external wave energy drive, the cascade of this energy ultimately leads to a balance between source and heating in the dissipative range, a stationary wave energy spectrum can build up even in the absence of significant nonlinear interactions among the waves. A phenomenon which is absent in a constant mean magnetic field. Assuming the existence of a conservative range, we determined the wave-energy spectrum for the special case of wave packets propagating in a "fully" chaotic magnetic field. It means that the distribution of "Lyapunov exponents", representing the rate of local exponentiation of nearby field lines, and hence the degree of stretching of wave packets, is strongly peaked at an average value α. This situation, which was first considered by Similon and Sudan (1989), for dissipation of Alfvén waves corresponds to chaotic mixing by flows with uniformly distributed rate of strain (Batchelor 1959, Antonsen et al. 1996).

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